

First exit time from a bounded interval for pseudo-processes driven by the equation

$$\partial/\partial t = (-1)^{N-1} \partial^{2N}/\partial x^{2N}$$

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Abstract

Let N be an integer greater than 1. We consider the pseudo-process $X = (X_t)_{t \geq 0}$ driven by the high-order heat-type equation $\partial/\partial t = (-1)^{N-1} \partial^{2N}/\partial x^{2N}$. Let us introduce the first exit time τ_{ab} from a bounded interval (a, b) by X ($a, b \in \mathbb{R}$) together with the related location, namely $X_{\tau_{ab}}$.

In this paper, we provide a representation of the joint pseudo-distribution of the vector $(\tau_{ab}, X_{\tau_{ab}})$ by means of some determinants. The method we use is based on a Feynman-Kac-like functional related to the pseudo-process X which leads to a boundary value problem. In particular, the pseudo-distribution of $X_{\tau_{ab}}$ admits a fine expression involving famous Hermite interpolating polynomials.

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1 Introduction

Let N be an integer greater than 1 and set $\kappa_N = (-1)^{N-1}$. We consider the pseudo-process $(X_t)_{t \geq 0}$ driven by the high-order heat-type equation $\partial/\partial t = \kappa_N \partial^{2N}/\partial x^{2N}$, the so-called pseudo-Brownian motion. This is the pseudo-Markov process with independent and stationary increments, associated with the *signed* heat-type kernel $p(t; x)$ which is the elementary solution of the foregoing equation. The kernel $p(t; x)$ is characterized by its Fourier transform:

$$\int_{-\infty}^{+\infty} e^{iux} p(t; x) dx = e^{-tu^{2N}}.$$

We define the related transition kernel as $p(t; x, y) = p(t; x - y)$ for any time $t > 0$ and any real numbers x, y , which represents the pseudo-probability that the pseudo-process started at x is in state y at time t . In symbols,

$$\mathbb{P}_x\{X_t \in dy\} = p(t; x, y) dy.$$

The \mathbb{P}_x , $x \in \mathbb{R}$, define a family of *signed* measures whose total mass equals one:

$$\mathbb{P}_x\{X_t \in \mathbb{R}\} = \int_{-\infty}^{+\infty} p(t; x, y) dy = 1.$$

The transition kernel $p(t; x, y)$ satisfies the backward and forward Kolmogorov equations

$$\frac{\partial p}{\partial t}(t; x, y) = \kappa_N \frac{\partial^{2N} p}{\partial x^{2N}}(t; x, y) = \kappa_N \frac{\partial^{2N} p}{\partial y^{2N}}(t; x, y).$$

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To be more precise, let us recall that the pseudo-Markov process $(X_t)_{t \geq 0}$ is defined according to the usual chain rule: for any positive integer n , for any times t_1, \dots, t_n such that $0 < t_1 < \dots < t_n$ and any real numbers x_1, \dots, x_n , and setting $t_0 = 0, x_0 = x$,

$$\mathbb{P}_x\{X_{t_1} \in dx_1, \dots, X_{t_n} \in dx_n\} = \left(\prod_{k=1}^n p(t_k - t_{k-1}; x_{k-1}, x_k) \right) dx_1 \dots dx_n. \quad (1.1)$$

In particular, by setting $T_t \phi(x) = \mathbb{E}_x[\phi(X_t)]$ for any time t , any real number x and any bounded \mathcal{C}^{2N} -function ϕ , the family $(T_t)_{t \geq 0}$ is a semi-group of operators whose infinitesimal generator \mathcal{G} is given by

$$\mathcal{G}\phi(x) = \lim_{h \rightarrow 0^+} \frac{1}{h} [\mathbb{E}_x[\phi(X_h)] - \phi(x)] = \kappa_N \phi^{(2N)}(x). \quad (1.2)$$

Above and throughout the paper, for any non-negative integer ℓ , $\phi^{(\ell)}$ stands for the derivative of ϕ of order ℓ .

The very notion of pseudo-process in a general framework goes back to Daletskii and Fomin in 1965 ([7]). The reader can find an extensive literature on the particular case of pseudo-Brownian motion. For instance, let us quote the works of Beghin, Cammarota, Hochberg, Krylov, Lachal, Nakajima, Nikitin, Nishioka, Orsingher, Ragozina ([2] to [6], [8, 9], [11] to [23]) and the references therein. These papers deal with several functionals related to pseudo-Brownian motion: sojourn time in a bounded or not interval, first overshooting time of a *single* level, maximum or minimum up to a fixed time... Let us mention also other interesting works : one dealing with high-order Schrödinger-type equation $\partial/\partial t = i\partial^{2N}/\partial x^{2N}$ which is related to the so-called Feynman-Kac measure [1], as well as [10] in which the authors develop an alternative and more probabilistic approach to pseudo-processes.

In [13, 14], we obtained the pseudo-distribution of the first overshooting time of a single threshold, together with the corresponding location at this time. In symbols, if τ_a denotes the first overshooting time of a fixed level a (upwards or downwards), we derived the joint pseudo-distribution of the couple (τ_a, X_{τ_a}) . Therein, we used an extension of famous Spitzer's identity. In [13, 14] and, in the particular case $N = 2$, in [20, 21], the authors observed a curious fact concerning the pseudo-distribution of X_{τ_a} : it is a linear combination of the Dirac distribution and its successive derivatives (in the sense of Schwartz distributions):

$$\mathbb{P}_x\{X_{\tau_a} \in dz\}/dz = \sum_{k=0}^{N-1} \frac{(a-x)^k}{k!} \delta_a^{(k)}(z). \quad (1.3)$$

The quantity $\delta_a^{(k)}$ is to be understood as the functional acting on test functions ϕ according as $\langle \delta_a^{(k)}, \phi \rangle = (-1)^k \phi^{(k)}(a)$. Formula (1.3) says that the overshoot through level a should be actually concentrated at a . The appearance of the Schwartz-Dirac distribution δ_a together with its successive derivatives can be interpreted by means of “multipoles” in reference to electric dipoles as in [13, 14] and, for $N = 2$, in [20, 21]. In particular, therein, δ_a and δ'_a are respectively named “monopole” and “dipole”. We refer the reader to [22] for a detailed account on monopoles and dipoles. An explanation of this curious fact should be found in considering a linear pseudo-random walk with $2N$ consecutive neighbours around each sites. Indeed, after suitably normalizing such a walk, the neighbours cluster into a single site and form a multipole; see the draft [17].

Till now, the first exit time from a bounded interval, or, equivalently, the first overshooting time of a *double* threshold has not yet been considered. This is the purpose of this work.

Let us introduce the first exit time from (a, b) (a, b being real numbers such that $a < b$) for $(X_t)_{t \geq 0}$:

$$\tau_{ab} = \inf\{t \geq 0 : X_t \notin (a, b)\}$$

with the usual convention that $\inf \emptyset = +\infty$. In this paper, we tackle the problem of finding the pseudo-distribution related to the double threshold: we provide a representation for the joint pseudo-distribution of the couple $(\tau_{ab}, X_{\tau_{ab}})$. This representation involves some determinants; this is the object of Theorems 1 and 2. For the location $X_{\tau_{ab}}$, we have the following counterpart to (1.3); see Theorem 4:

$$\mathbb{P}_x\{X_{\tau_{ab}} \in dz\}/dz = \sum_{k=0}^{N-1} (-1)^k H_k^-(x) \delta_a^{(k)}(z) + \sum_{k=0}^{N-1} (-1)^k H_k^+(x) \delta_b^{(k)}(z) \quad (1.4)$$

where the functions H_k^\pm , $0 \leq k \leq N-1$, are the classical Hermite interpolating polynomials of degree $(2N-1)$ related to points a and b satisfying

$$(H_k^-)^{(\ell)}(a) = (H_k^+)^{(\ell)}(b) = \delta_{k\ell}, \quad (H_k^+)^{(\ell)}(a) = (H_k^-)^{(\ell)}(b) = 0, \quad 0 \leq \ell \leq N-1.$$

Above, the quantity $\delta_{k\ell}$ denotes the usual Kronecker symbol: if $k = \ell$, $\delta_{k\ell} = 1$, else $\delta_{k\ell} = 0$. They explicitly write as

$$H_k^-(x) = \left(\frac{b-x}{b-a}\right)^N \frac{(x-a)^k}{k!} \sum_{\ell=0}^{N-k-1} \binom{\ell+N-1}{\ell} \left(\frac{x-a}{b-a}\right)^\ell,$$

$$H_k^+(x) = \left(\frac{x-a}{b-a}\right)^N \frac{(x-b)^k}{k!} \sum_{\ell=0}^{N-k-1} \binom{\ell+N-1}{\ell} \left(\frac{b-x}{b-a}\right)^\ell.$$

In particular, we can deduce from (1.4) the “ruin pseudo-probabilities”, that is, the pseudo-probabilities of overshooting one level (a or b) before the other one; see Corollary 3.

These results have been announced without any proof in a survey on pseudo-Brownian motion, [16], after a conference held in Madrid (IWAP 2010).

Throughout the paper, the function φ denotes any $(N-1)$ times differentiable function.

2 Feynman-Kac functional

We start from the following fact: in [13, 14], we first obtained the pseudo-distribution of the couple $(\sup_{0 \leq s \leq t} X_s, X_t)$ by making use of an extension of Spitzer’s identity. From this, we deduced that of the couple (τ_a, X_{τ_a}) and we made the observation that, for any $\lambda \geq 0$ and any $(N-1)$ times differentiable bounded function φ , the Feynman-Kac functional $\Phi(x) = \mathbb{E}_x(e^{-\lambda \tau_a} \varphi(X_{\tau_a}) \mathbf{1}_{\{\tau_a < +\infty\}})$ solves the boundary value problem

$$\begin{cases} \kappa_N \Phi^{(2N)}(x) = \lambda \Phi(x), & x \in (-\infty, a) \text{ (or } x \in (a, +\infty)), \\ \Phi^{(k)}(a) = \varphi^{(k)}(a) & \text{for } k \in \{0, 1, \dots, N-1\}. \end{cases} \quad (2.1)$$

So, we state the heuristic that an analogous boundary value problem should hold for the Feynman-Kac functional related to τ_{ab} . The results obtained here through this approach coincide with limiting results deduced from a suitable pseudo-random walk studied in [17]. Moreover, when taking the limit as a goes to $-\infty$ or b goes to $+\infty$ in the present results, we retrieve the pseudo-distribution of (τ_a, X_{τ_a}) obtained in [14]. So, these observations comfort us in our heuristic. Actually, our purpose in this work is essentially concentrated in calculating the pseudo-distribution of $(\tau_{ab}, X_{\tau_{ab}})$.

As pointed out in several works on pseudo-processes, pseudo-Brownian motion is properly defined only on the set of dyadic times and ad-hoc definitions should be taken for computing certain functionals of this pseudo-process depending on a continuous set of times; see, e.g., [14] and, in the particular case $N = 2$, [21]. Roughly speaking, the dense subset of dyadic times is appropriate because of the usual property that for any $n \in \mathbb{N}$, $\{k/2^n, k \in \mathbb{N}\} \subset \{k/2^{n+1}, k \in \mathbb{N}\}$. Indeed, this latter permits to view the pseudo-process $(X_t)_{t \geq 0}$ as an informal limit of the family of step-processes $(X_{n,t})_{t \geq 0}$ defined according to the following sampling procedure:

$$X_{n,t} = \sum_{k=0}^{\infty} \mathbb{1}_{[k/2^n, (k+1)/2^n)}(t) X_{k/2^n}.$$

For each fixed $n \in \mathbb{N}$, the sequence $(X_{k/2^n})_{k \in \mathbb{N}}$ can be correctly defined thanks to (1.1). But the fact that $\int_{-\infty}^{+\infty} |p(t; x)| dx > 1$ prevent us from applying the classical extension theorem of Kolmogorov for finding *a priori* a σ -additive measure on the usual space of right-continuous functions on $[0, +\infty)$ which have left-hand limits, measure whose finite projections would yield the finite-dimensional pseudo-distributions of the sequence $(X_{k/2^n})_{k \in \mathbb{N}}$.

For our concern, we set

$$\tau_{ab,n} = \frac{1}{2^n} \min\{k \in \mathbb{N} : X_{k/2^n} \notin (a, b)\}$$

and, for $x \in (a, b)$,

$$\Phi_n(x) = \mathbb{E}_x \left(e^{-\lambda \tau_{ab,n}} \varphi(X_{\tau_{ab,n}}) \mathbf{1}_{\{\tau_{ab,n} < +\infty\}} \right).$$

Then, we define the Feynman-Kac functional $\Phi(x) = \mathbb{E}_x \left(e^{-\lambda \tau_{ab}} \varphi(X_{\tau_{ab}}) \mathbf{1}_{\{\tau_{ab} < +\infty\}} \right)$ as the limit

$$\Phi(x) \stackrel{\text{def}}{=} \lim_{n \rightarrow +\infty} \Phi_n(x)$$

and we state below the analogue to (2.1).

Heuristic. For any $\lambda \geq 0$ and any $(N-1)$ times differentiable bounded function φ , the Feynman-Kac functional $\Phi(x) = \mathbb{E}_x \left(e^{-\lambda \tau_{ab}} \varphi(X_{\tau_{ab}}) \mathbf{1}_{\{\tau_{ab} < +\infty\}} \right)$ solves the boundary value problem

$$\begin{cases} \kappa_N \Phi^{(2N)}(x) = \lambda \Phi(x), & x \in (a, b), \\ \Phi^{(k)}(a) = \varphi^{(k)}(a) \quad \text{and} \quad \Phi^{(k)}(b) = \varphi^{(k)}(b) & \text{for } k \in \{0, 1, \dots, N-1\}. \end{cases} \quad (2.2)$$

3 Joint pseudo-distribution of $(\tau_{ab}, X_{\tau_{ab}})$

In this section, we solve boundary value problem (2.2) in order to derive the joint pseudo-probability of $(\tau_{ab}, X_{\tau_{ab}})$. In this way, if we choose $\varphi(x) = e^{i\mu x}$, $\mu \in \mathbb{R}$, we first obtain its Laplace-Fourier transform. Actually, the results we derived hold true for any $(N-1)$ times differentiable function φ .

Let us introduce the $(2N)$ th roots of κ_N : $\theta_\ell = e^{i\frac{2\ell+N-1}{2N}\pi}$, $1 \leq \ell \leq 2N$. We have $\theta_\ell^{2N} = \kappa_N$. For any complex number z , we set $e_\lambda^z = e^{\lambda^{1/(2N)}z}$.

Theorem 1. The Feynman-Kac functional related to $(\tau_{ab}, X_{\tau_{ab}})$ admits the following representation:

$$\mathbb{E}_x \left(e^{-\lambda \tau_{ab}} \varphi(X_{\tau_{ab}}) \mathbf{1}_{\{\tau_{ab} < +\infty\}} \right) = \sum_{k=0}^{N-1} \lambda^{-\frac{k}{2N}} \frac{\Delta_k^-(\lambda; x)}{\Delta(\lambda)} \varphi^{(k)}(a) + \sum_{k=0}^{N-1} \lambda^{-\frac{k}{2N}} \frac{\Delta_k^+(\lambda; x)}{\Delta(\lambda)} \varphi^{(k)}(b) \quad (3.1)$$

where the quantities $\Delta(\lambda)$ and $\Delta_k^\pm(\lambda; x)$ are the determinants below:

$$\Delta(\lambda) = \begin{vmatrix} e_\lambda^{\theta_1 a} & \dots & e_\lambda^{\theta_{2N} a} \\ \theta_1 e_\lambda^{\theta_1 a} & \dots & \theta_{2N} e_\lambda^{\theta_{2N} a} \\ \vdots & & \vdots \\ \theta_1^{N-1} e_\lambda^{\theta_1 a} & \dots & \theta_{2N}^{N-1} e_\lambda^{\theta_{2N} a} \\ \dots & \dots & \dots \\ e_\lambda^{\theta_1 b} & \dots & e_\lambda^{\theta_{2N} b} \\ \theta_1 e_\lambda^{\theta_1 b} & \dots & \theta_{2N} e_\lambda^{\theta_{2N} b} \\ \vdots & & \vdots \\ \theta_1^{N-1} e_\lambda^{\theta_1 b} & \dots & \theta_{2N}^{N-1} e_\lambda^{\theta_{2N} b} \end{vmatrix}$$

and

$$\Delta_k^-(\lambda; x) = \begin{vmatrix} e_\lambda^{\theta_1 a} & \cdots & e_\lambda^{\theta_{2N} a} \\ \vdots & & \vdots \\ \theta_1^{k-1} e_\lambda^{\theta_1 a} & \cdots & \theta_{2N}^{k-1} e_\lambda^{\theta_{2N} a} \\ e_\lambda^{\theta_1 x} & \cdots & e_\lambda^{\theta_{2N} x} \\ \theta_1^{k+1} e_\lambda^{\theta_1 a} & \cdots & \theta_{2N}^{k+1} e_\lambda^{\theta_{2N} a} \\ \vdots & & \vdots \\ \theta_1^{N-1} e_\lambda^{\theta_1 a} & \cdots & \theta_{2N}^{N-1} e_\lambda^{\theta_{2N} a} \\ \cdots & & \cdots \\ e_\lambda^{\theta_1 b} & \cdots & e_\lambda^{\theta_{2N} b} \\ \vdots & & \vdots \\ \theta_1^{N-1} e_\lambda^{\theta_1 b} & \cdots & \theta_{2N}^{N-1} e_\lambda^{\theta_{2N} b} \end{vmatrix}, \quad \Delta_k^+(\lambda; x) = \begin{vmatrix} e_\lambda^{\theta_1 a} & \cdots & e_\lambda^{\theta_{2N} a} \\ \vdots & & \vdots \\ \theta_1^{N-1} e_\lambda^{\theta_1 a} & \cdots & \theta_{2N}^{N-1} e_\lambda^{\theta_{2N} a} \\ \cdots & & \cdots \\ e_\lambda^{\theta_1 b} & \cdots & e_\lambda^{\theta_{2N} b} \\ \vdots & & \vdots \\ \theta_1^{k-1} e_\lambda^{\theta_1 b} & \cdots & \theta_{2N}^{k-1} e_\lambda^{\theta_{2N} b} \\ e_\lambda^{\theta_1 x} & \cdots & e_\lambda^{\theta_{2N} x} \\ \theta_1^{k+1} e_\lambda^{\theta_1 b} & \cdots & \theta_{2N}^{k+1} e_\lambda^{\theta_{2N} b} \\ \vdots & & \vdots \\ \theta_1^{N-1} e_\lambda^{\theta_1 b} & \cdots & \theta_{2N}^{N-1} e_\lambda^{\theta_{2N} b} \end{vmatrix}.$$

The functions $x \mapsto \Delta_k^\pm(\lambda; x)$, $0 \leq k \leq N-1$, are the solutions of the boundary value problems

$$\begin{cases} (\Delta_k^-)^{(2N)}(\lambda; x) = \kappa_N \lambda \Delta_k^-(\lambda; x), \\ (\Delta_k^-)^{(\ell)}(\lambda; a) = \delta_{k\ell} \lambda^{\ell/(2N)} \Delta(\lambda), \quad (\Delta_k^-)^{(\ell)}(\lambda; b) = 0 \quad \text{for } \ell \in \{0, \dots, N-1\}, \end{cases}$$

$$\begin{cases} (\Delta_k^+)^{(2N)}(\lambda; x) = \kappa_N \lambda \Delta_k^+(\lambda; x), \\ (\Delta_k^+)^{(\ell)}(\lambda; a) = 0, \quad (\Delta_k^+)^{(\ell)}(\lambda; b) = \delta_{k\ell} \lambda^{\ell/(2N)} \Delta(\lambda) \quad \text{for } \ell \in \{0, \dots, N-1\}. \end{cases}$$

Proof. The solution of linear boundary value problem (2.2) has the form $\Phi(x) = \sum_{\ell=1}^{2N} \alpha_\ell e_\lambda^{\theta_\ell x}$ where the coefficients α_ℓ , $1 \leq \ell \leq 2N$, satisfy the linear system below:

$$\begin{cases} \sum_{\ell=1}^{2N} \theta_\ell^k e_\lambda^{\theta_\ell a} \alpha_\ell = \lambda^{-\frac{k}{2N}} \varphi^{(k)}(a), \quad 0 \leq k \leq N-1, \\ \sum_{\ell=1}^{2N} \theta_\ell^k e_\lambda^{\theta_\ell b} \alpha_\ell = \lambda^{-\frac{k}{2N}} \varphi^{(k)}(b), \quad 0 \leq k \leq N-1. \end{cases} \quad (3.2)$$

This system can be solved by using Cramer's formulae:

$$\alpha_\ell = \frac{\Delta_\ell(\lambda, \varphi)}{\Delta(\lambda)}, \quad 1 \leq \ell \leq 2N,$$

where $\Delta(\lambda)$ is the determinant displayed in Theorem 1 and $\Delta_\ell(\lambda, \varphi)$ is the determinant deduced from $\Delta(\lambda)$ by replacing its ℓ th column by the right-hand side of (3.2), that is

$$\Delta_\ell(\lambda, \varphi) = \begin{vmatrix} e_\lambda^{\theta_1 a} & \cdots & e_\lambda^{\theta_{\ell-1} a} & \varphi(a) & e_\lambda^{\theta_{\ell+1} a} & \cdots & e_\lambda^{\theta_{2N} a} \\ \theta_1 e_\lambda^{\theta_1 a} & \cdots & \theta_{\ell-1} e_\lambda^{\theta_{\ell-1} a} & \lambda^{-\frac{1}{2N}} \varphi'(a) & \theta_{\ell+1} e_\lambda^{\theta_{\ell+1} a} & \cdots & \theta_{2N} e_\lambda^{\theta_{2N} a} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \theta_1^{N-1} e_\lambda^{\theta_1 a} & \cdots & \theta_{\ell-1}^{N-1} e_\lambda^{\theta_{\ell-1} a} & \lambda^{-\frac{N-1}{2N}} \varphi^{(N-1)}(a) & \theta_{\ell+1}^{N-1} e_\lambda^{\theta_{\ell+1} a} & \cdots & \theta_{2N}^{N-1} e_\lambda^{\theta_{2N} a} \\ \cdots & & \cdots & \cdots & \cdots & & \cdots \\ e_\lambda^{\theta_1 b} & \cdots & e_\lambda^{\theta_{\ell-1} b} & \varphi(b) & e_\lambda^{\theta_{\ell+1} b} & \cdots & e_\lambda^{\theta_{2N} b} \\ \theta_1 e_\lambda^{\theta_1 b} & \cdots & \theta_{\ell-1} e_\lambda^{\theta_{\ell-1} b} & \lambda^{-\frac{1}{2N}} \varphi'(b) & \theta_{\ell+1} e_\lambda^{\theta_{\ell+1} b} & \cdots & \theta_{2N} e_\lambda^{\theta_{2N} b} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \theta_1^{N-1} e_\lambda^{\theta_1 b} & \cdots & \theta_{\ell-1}^{N-1} e_\lambda^{\theta_{\ell-1} b} & \lambda^{-\frac{N-1}{2N}} \varphi^{(N-1)}(b) & \theta_{\ell+1}^{N-1} e_\lambda^{\theta_{\ell+1} b} & \cdots & \theta_{2N}^{N-1} e_\lambda^{\theta_{2N} b} \end{vmatrix}.$$

The determinant $\Delta_\ell(\lambda, \varphi)$ can be expanded with respect to its ℓ th column:

$$\Delta_\ell(\lambda, \varphi) = \sum_{k=0}^{N-1} \lambda^{-\frac{k}{2N}} \Delta_{k\ell}^-(\lambda) \varphi^{(k)}(a) + \sum_{k=0}^{N-1} \lambda^{-\frac{k}{2N}} \Delta_{k\ell}^+(\lambda) \varphi^{(k)}(b)$$

with

[illegible]

and, in the same manner,

$$\Delta_{k\ell}^+(\lambda) = \begin{vmatrix} e_{\lambda}^{\theta_1 a} & \dots & e_{\lambda}^{\theta_{\ell-1} a} & e_{\lambda}^{\theta_{\ell} a} & e_{\lambda}^{\theta_{\ell+1} a} & \dots & e_{\lambda}^{\theta_{2N} a} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \theta_1^{N-1} e_{\lambda}^{\theta_1 a} & \dots & \theta_{\ell-1}^{N-1} e_{\lambda}^{\theta_{\ell-1} a} & \theta_{\ell}^{N-1} e_{\lambda}^{\theta_{\ell} a} & \theta_{\ell+1}^{N-1} e_{\lambda}^{\theta_{\ell+1} a} & \dots & \theta_{2N}^{N-1} e_{\lambda}^{\theta_{2N} a} \\ \hline e_{\lambda}^{\theta_1 b} & \dots & e_{\lambda}^{\theta_{\ell-1} b} & e_{\lambda}^{\theta_{\ell} b} & e_{\lambda}^{\theta_{\ell+1} b} & \dots & e_{\lambda}^{\theta_{2N} b} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \theta_1^{k-1} e_{\lambda}^{\theta_1 b} & \dots & \theta_{\ell-1}^{k-1} e_{\lambda}^{\theta_{\ell-1} b} & \theta_{\ell}^{k-1} e_{\lambda}^{\theta_{\ell} b} & \theta_{\ell+1}^{k-1} e_{\lambda}^{\theta_{\ell+1} b} & \dots & \theta_{2N}^{k-1} e_{\lambda}^{\theta_{2N} b} \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ \theta_1^{k+1} e_{\lambda}^{\theta_1 b} & \dots & \theta_{\ell-1}^{k+1} e_{\lambda}^{\theta_{\ell-1} b} & \theta_{\ell}^{k+1} e_{\lambda}^{\theta_{\ell} b} & \theta_{\ell+1}^{k+1} e_{\lambda}^{\theta_{\ell+1} b} & \dots & \theta_{2N}^{k+1} e_{\lambda}^{\theta_{2N} b} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \theta_1^{N-1} e_{\lambda}^{\theta_1 b} & \dots & \theta_{\ell-1}^{N-1} e_{\lambda}^{\theta_{\ell-1} b} & \theta_{\ell}^{N-1} e_{\lambda}^{\theta_{\ell} b} & \theta_{\ell+1}^{N-1} e_{\lambda}^{\theta_{\ell+1} b} & \dots & \theta_{2N}^{N-1} e_{\lambda}^{\theta_{2N} b} \end{vmatrix}.$$

With these settings at hand, we can write the solution of (2.2):

$$\begin{aligned} \Phi(x) &= \sum_{\ell=1}^{2N} \alpha_{\ell} e_{\lambda}^{\theta_{\ell} x} = \sum_{\ell=1}^{2N} \frac{\Delta_{\ell}(\lambda, \varphi)}{\Delta(\lambda)} e_{\lambda}^{\theta_{\ell} x} \\ &= \frac{1}{\Delta(\lambda)} \left[\sum_{k=0}^{N-1} \lambda^{-\frac{k}{2N}} \left(\sum_{\ell=1}^{2N} \Delta_{k\ell}^{-}(\lambda) e_{\lambda}^{\theta_{\ell} x} \right) \varphi^{(k)}(a) + \sum_{k=0}^{N-1} \lambda^{-\frac{k}{2N}} \left(\sum_{\ell=1}^{2N} \Delta_{k\ell}^{+}(\lambda) e_{\lambda}^{\theta_{\ell} x} \right) \varphi^{(k)}(b) \right] \\ &= \sum_{k=0}^{N-1} \lambda^{-\frac{k}{2N}} \frac{\Delta_k^{-}(\lambda; x)}{\Delta(\lambda)} \varphi^{(k)}(a) + \sum_{k=0}^{N-1} \lambda^{-\frac{k}{2N}} \frac{\Delta_k^{+}(\lambda; x)}{\Delta(\lambda)} \varphi^{(k)}(b) \end{aligned}$$

with

$$\Delta_k^{-}(\lambda; x) = \sum_{\ell=1}^{2N} \Delta_{k\ell}^{-}(\lambda) e_{\lambda}^{\theta_{\ell} x}, \quad \Delta_k^{+}(\lambda; x) = \sum_{\ell=1}^{2N} \Delta_{k\ell}^{+}(\lambda) e_{\lambda}^{\theta_{\ell} x}. \quad (3.3)$$

We immediately see that equalities (3.3) are the expansions of the determinants displayed in Theorem 1 with respect to their $(k-1)$ th row and $(k+N-1)$ th row respectively. Formula (3.1) is proved.

Finally, it is easy to check the boundary value problems satisfied by the functions $x \mapsto \Delta_k^{\pm}(\lambda; x)$ by using elementary rules on differentiating a determinant. In particular for, e.g., Δ_k^{-} , the determinants defining $(\Delta_k^{-})^{(\ell)}(\lambda; a)$, $\ell \in \{0, \dots, N-1\} \setminus \{k\}$, and $(\Delta_k^{-})^{(\ell)}(\lambda; b)$, $\ell \in \{0, \dots, N-1\}$, have two identical rows, thus they vanish. The determinant $(\Delta_k^{-})^{(k)}(\lambda; a)$ has the same rows as $\Delta(\lambda)$ up to the multiplicative factor $\lambda^{k/(2N)}$ for its k th row, then it coincides with $\lambda^{k/(2N)} \Delta(\lambda)$. The proof of Theorem 1 is finished. \square

Now, by eliminating the function φ in (3.1), we get the following result which should be understood in the sense of Schwartz distributions:

$$\begin{aligned} \mathbb{E}_x \left(e^{-\lambda \tau_{ab}} \mathbf{1}_{\{\tau_{ab} < +\infty\}}, X_{\tau_{ab}} \in dz \right) / dz \\ = \sum_{k=0}^{N-1} (-1)^k \lambda^{-\frac{k}{2N}} \frac{\Delta_k^{-}(\lambda; x)}{\Delta(\lambda)} \delta_a^{(k)}(z) + \sum_{k=0}^{N-1} (-1)^k \lambda^{-\frac{k}{2N}} \frac{\Delta_k^{+}(\lambda; x)}{\Delta(\lambda)} \delta_b^{(k)}(z) \end{aligned} \quad (3.4)$$

from which we derive the following representation for the pseudo-distribution of $(\tau_{ab}, X_{\tau_{ab}})$.

Theorem 2. *The joint pseudo-distribution of $(\tau_{ab}, X_{\tau_{ab}})$ admits the following representation:*

$$\mathbb{P}_x \{ \tau_{ab} \in dt, X_{\tau_{ab}} \in dz \} / dt dz = \sum_{k=0}^{N-1} (-1)^k I_k^{-}(t; x) \delta_a^{(k)}(z) + \sum_{k=0}^{N-1} (-1)^k I_k^{+}(t; x) \delta_b^{(k)}(z) \quad (3.5)$$

where the functions $I_k^{\pm}(t; x)$, $0 \leq k \leq N-1$, are characterized by their Laplace transforms:

$$\int_0^{\infty} I_k^{\pm}(t; x) e^{-\lambda t} dt = \lambda^{-\frac{k}{2N}} \frac{\Delta_k^{\pm}(\lambda; x)}{\Delta(\lambda)}.$$

They are also characterized by the boundary value problems

$$\begin{cases} \frac{\partial I_k^-}{\partial t}(t; x) = \kappa_N \frac{\partial^{2N} I_k^-}{\partial^{2N} x}(t; x) \\ \frac{\partial^\ell I_k^-}{\partial^\ell x}(t; a) = \delta_{k\ell}, \quad \frac{\partial^\ell I_k^-}{\partial^\ell x}(t; b) = 0 \quad \text{for } \ell \in \{0, 1, \dots, N-1\}, \\ \frac{\partial I_k^+}{\partial t}(t; x) = \kappa_N \frac{\partial^{2N} I_k^+}{\partial^{2N} x}(t; x) \\ \frac{\partial^\ell I_k^+}{\partial^\ell x}(t; a) = 0, \quad \frac{\partial^\ell I_k^+}{\partial^\ell x}(t; b) = \delta_{k\ell} \quad \text{for } \ell \in \{0, 1, \dots, N-1\}. \end{cases}$$

The boundary value problems satisfied by the functions I_k^\pm , $0 \leq k \leq N-1$, come from those satisfied by the functions Δ_k^\pm displayed in Theorem 1. The only details we have to check are that $I_k^\pm(t; x)$ goes to 0 as t tends to 0^+ and that $I_k^\pm(t; x)$ is bounded as t tends to $+\infty$ (in order to have $\int_0^\infty (\partial/\partial t) I_k^\pm(t; x) e^{-\lambda t} dt = \lambda \int_0^\infty I_k^\pm(t; x) e^{-\lambda t} dt$) which can be deduced from the fact that their Laplace transforms go to 0 exponentially quickly as λ goes to $+\infty$ and are bounded as λ goes to 0^+ . These facts are proved in Appendix A; see (A.2) and (A.5).

Remark 1. The functions I_k^\pm , $0 \leq k \leq N-1$, are real-valued. Indeed, observing that the complex numbers θ_ℓ , $1 \leq \ell \leq 2N$, are conjugate two by two, it is easily seen that the determinants contain conjugate columns two by two, so they are real numbers. More precisely, conjugating $\theta_1, \dots, \theta_N, \theta_{N+1}, \dots, \theta_{2N}$ respectively yields $\theta_N, \dots, \theta_1, \theta_{2N}, \dots, \theta_{N+1}$. Therefore, conjugating the determinants Δ and $\Delta_{k\ell}^\pm$ boils down to interchanging their 1st and N th columns, their 2nd and $(N-1)$ th columns, \dots , their $(N+1)$ th and $(2N)$ th columns, their $(N+2)$ th and $(2N-1)$ th columns, and so on. In this way, we perform an even number of transpositions and we retrieve the original determinants: $\overline{\Delta} = \Delta$ and $\overline{\Delta_{k\ell}^\pm} = \Delta_{k\ell}^\pm$, proving that they are real numbers.

Moreover, the functions I_k^+ and I_k^- are related according to the identity $I_k^+(t; x) = (-1)^k I_k^-(t; a + b - x)$ as it can be seen by proving the same identity concerning their Laplace transforms; see (A.1) in Appendix A.

Remark 2. Let us compute the limit of (3.4) as b tends towards $+\infty$. To this aim, we find that

$$\frac{\Delta_k^-(\lambda; x)}{\Delta(\lambda)} \xrightarrow{b \rightarrow +\infty} \sum_{\ell=1}^N \alpha_{k\ell} e^{\theta_\ell(x-a)} \quad \text{and} \quad \frac{\Delta_k^+(\lambda; x)}{\Delta(\lambda)} \xrightarrow{b \rightarrow +\infty} 0 \quad (3.6)$$

with

$$\alpha_{k\ell} = \frac{1}{\det(V)} \begin{vmatrix} 1 & \dots & 1 & 0 & 1 & \dots & 1 \\ \theta_1 & \dots & \theta_{\ell-1} & 0 & \theta_{\ell+1} & \dots & \theta_N \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \theta_1^{k-1} & \dots & \theta_{\ell-1}^{k-1} & 0 & \theta_{\ell+1}^{k-1} & \dots & \theta_N^{k-1} \\ 0 & \dots & 0 & 1 & 0 & \dots & 0 \\ \theta_1^{k+1} & \dots & \theta_{\ell-1}^{k+1} & 0 & \theta_{\ell+1}^{k+1} & \dots & \theta_N^{k+1} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \theta_1^{N-1} & \dots & \theta_{\ell-1}^{N-1} & 0 & \theta_{\ell+1}^{N-1} & \dots & \theta_N^{N-1} \end{vmatrix}.$$

The coefficients $\alpha_{k\ell}$ are characterized by the identity

$$\sum_{k=0}^{N-1} \alpha_{k\ell} x^k = \frac{1}{\det(V)} \begin{vmatrix} 1 & \dots & 1 & 1 & 1 & \dots & 1 \\ \theta_1 & \dots & \theta_{\ell-1} & x & \theta_{\ell+1} & \dots & \theta_N \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \theta_1^{N-1} & \dots & \theta_{\ell-1}^{N-1} & x^{N-1} & \theta_{\ell+1}^{N-1} & \dots & \theta_N^{N-1} \end{vmatrix} = \prod_{\substack{1 \leq k \leq N \\ k \neq \ell}} \left(\frac{x - \theta_k}{\theta_\ell - \theta_k} \right) \quad (3.7)$$

as it is easily seen by appealing to the well-known Vandermonde determinant $\det(V) = \prod_{1 \leq i < j \leq N} (\theta_j - \theta_i)$. Notice that polynomial (3.7) is nothing but an elementary Lagrange interpolating polynomial related to the numbers θ_i , $1 \leq i \leq N$. The details of these limiting results being cumbersome, we postpone them to Appendix A.

In regards to (3.4), (3.5) and (3.6), we conclude that, for $x > a$,

$$\lim_{b \rightarrow +\infty} \mathbb{P}_x\{\tau_{ab} \in dt, X_{\tau_{ab}} \in dz\} / dt dz = \sum_{k=0}^{N-1} (-1)^k K_k(t; x) \delta_a^{(k)}(z)$$

where K_k is the function whose Laplace transform is given by

$$\int_0^\infty K_k(t; x) e^{-\lambda t} dt = \lambda^{-\frac{k}{2N}} \sum_{\ell=1}^N \alpha_{k\ell} e^{\theta_\ell \sqrt{2N\lambda} (x-a)}.$$

We retrieve at the limit the pseudo-distribution of (τ_a, X_{τ_a}) related to the first overshooting time of level a displayed in [14], formula (5.15).

4 Pseudo-distribution of τ_{ab}

By applying the Schwartz distribution (3.5) to the test function 1, we immediately extract the pseudo-distribution of τ_{ab} : $\mathbb{P}_x\{\tau_{ab} \in dt\} / dt = I_0^-(t; x) + I_0^+(t; x)$ that we state as follows.

Theorem 3. *The pseudo-distribution of τ_{ab} is given either by one of both formulae below:*

$$\mathbb{P}_x\{\tau_{ab} \in dt\} / dt = I(t; x), \quad \mathbb{P}_x\{\tau_{ab} \leq t\} = J(t; x)$$

with

$$\int_0^\infty I(t; x) e^{-\lambda t} dt = \frac{\Delta_0^+(\lambda; x) + \Delta_0^-(\lambda; x)}{\Delta(\lambda)}, \quad \int_0^\infty J(t; x) e^{-\lambda t} dt = \frac{1}{\lambda} \frac{\Delta_0^+(\lambda; x) + \Delta_0^-(\lambda; x)}{\Delta(\lambda)}.$$

Let us introduce the up-to-date minimum and maximum functionals of X :

$$m_t = \min_{s \in [0, t]} X_s, \quad M_t = \max_{s \in [0, t]} X_s.$$

It is plain that the functionals m_t, M_t and time τ_{ab} are related according as $a < m_t \leq M_t < b \iff \tau_{ab} > t$. Then $\mathbb{P}_x\{a < m_t \leq M_t < b\} = 1 - \mathbb{P}_x\{\tau_{ab} \leq t\}$.

Corollary 1. *The joint pseudo-distribution of (m_t, M_t) is given by*

$$\mathbb{P}_x\{a < m_t \leq M_t < b\} = 1 - J(t; x)$$

and its Laplace transform with respect to t writes

$$\int_0^\infty \mathbb{P}_x\{a < m_t \leq M_t < b\} e^{-\lambda t} dt = \frac{1}{\lambda} \frac{\Delta(\lambda) - \Delta_0^+(\lambda; x) - \Delta_0^-(\lambda; x)}{\Delta(\lambda)}.$$

5 Pseudo-distribution of $X_{\tau_{ab}}$

In this part, we focus on the exit location of X at time τ_{ab} whose pseudo-distribution admits a remarkable expression by means of Hermite interpolating polynomials whose expressions are displayed in the introduction.

Theorem 4. *The pseudo-distribution of the exit location $X_{\tau_{ab}} \mathbb{1}_{\{\tau_{ab} < +\infty\}}$ is given, in the sense of Schwartz distributions, by*

$$\mathbb{P}_x\{X_{\tau_{ab}} \in dz, \tau_{ab} < +\infty\} / dz = \sum_{k=0}^{N-1} (-1)^k H_k^-(x) \delta_a^{(k)}(z) + \sum_{k=0}^{N-1} (-1)^k H_k^+(x) \delta_b^{(k)}(z). \quad (5.1)$$

Proof. We directly solve boundary value problem (2.2) in the case where $\lambda = 0$ therein. Namely, by setting $\Psi(x) = \mathbb{E}_x\left(\varphi(X_{\tau_{ab}}) \mathbb{1}_{\{\tau_{ab} < +\infty\}}\right)$,

$$\begin{cases} \Psi^{(2N)}(x) = 0, & x \in (a, b), \\ \Psi^{(k)}(a) = \varphi^{(k)}(a) \text{ and } \Psi^{(k)}(b) = \varphi^{(k)}(b) & \text{for } k \in \{0, 1, \dots, N-1\}. \end{cases}$$

It is clear that Ψ is the polynomial of degree not greater than $(2N - 1)$ whose derivatives at a and b up to order $(N - 1)$ are the given numbers $\varphi^{(k)}(a)$ and $\varphi^{(k)}(b)$, $0 \leq k \leq N - 1$. It can be written as a linear combination of the Hermite interpolating fundamental polynomials H_k^\pm , $0 \leq k \leq N - 1$, displayed in Theorem 4 as follows: for any test functions φ ,

$$\mathbb{E}_x\left(\varphi(X_{\tau_{ab}}) \mathbf{1}_{\{\tau_{ab} < +\infty\}}\right) = \sum_{k=0}^{N-1} H_k^-(x) \varphi^{(k)}(a) + \sum_{k=0}^{N-1} H_k^+(x) \varphi^{(k)}(b). \quad (5.2)$$

Formula (5.1) is nothing but (5.2) rephrased by means of Schwartz distributions. \square

Remark 3. Formula (5.2) yields for $\varphi = H_k^\pm$, $0 \leq k \leq N - 1$, that $\mathbb{E}_x\left(H_k^\pm(X_{\tau_{ab}}) \mathbf{1}_{\{\tau_{ab} < +\infty\}}\right) = H_k^\pm(x)$.

Remark 4. By letting b tend to $+\infty$, we see that $H_k^+(x)$ tends to 0 while $H_k^-(x)$ tends to $(x - a)^k / (k!)$. Hence, we find that

$$\lim_{b \rightarrow +\infty} \mathbb{P}_x\{X_{\tau_{ab}} \in dz, \tau_{ab} < +\infty\} / dz = \sum_{k=0}^{N-1} \frac{(a - x)^k}{k!} \delta_a^{(k)}(z).$$

We retrieve at the limit the pseudo-distribution (1.3) of the location X_{τ_a} of X at the first overshooting time of level a , which is displayed in [14], formula (5.18).

Corollary 2. Time τ_{ab} is \mathbb{P}_x -almost surely finite in the sense that

$$\mathbb{P}_x\{\tau_{ab} < +\infty\} = 1.$$

Because of this, in the sequel of the paper, we shall omit the condition $\tau_{ab} < +\infty$ when considering the pseudo-random variable $X_{\tau_{ab}}$. Actually, let us recall that, in the framework of signed measures, if A is a set of \mathbb{P}_x -measure 1, it does *not* entail that for any set B that $\mathbb{P}_x(A \cap B) = \mathbb{P}_x(B)$ contrarily to the case of ordinary probability.

Proof. The pseudo-probability $\mathbb{P}\{\tau_{ab} < +\infty\}$ can be deduced from (5.1) by choosing $\varphi = 1$. Indeed, we have that

$$\begin{aligned} H_0^-(x) &= \left(\frac{b-x}{b-a}\right)^N \sum_{\ell=0}^{N-1} \binom{\ell+N-1}{\ell} \left(\frac{x-a}{b-a}\right)^\ell \\ &= \frac{(b-x)^N}{(b-a)^{2N-1}} \sum_{\ell=0}^{N-1} \binom{\ell+N-1}{\ell} (x-a)^\ell (b-a)^{N-1-\ell}. \end{aligned}$$

By writing the term $(b-a)^{N-1-\ell}$ as

$$(b-a)^{N-1-\ell} = [(x-a) + (b-x)]^{N-1-\ell} = \sum_{k=0}^{N-1-\ell} \binom{N-1-\ell}{k} (x-a)^k (b-x)^{N-1-k-\ell},$$

it follows that

$$\begin{aligned} H_0^-(x) &= \frac{1}{(b-a)^{2N-1}} \sum_{\substack{0 \leq \ell \leq N-1 \\ 0 \leq k \leq N-1-\ell}} \binom{N-1-\ell}{k} \binom{\ell+N-1}{\ell} (x-a)^{k+\ell} (b-x)^{2N-1-k-\ell} \\ &= \frac{1}{(b-a)^{2N-1}} \sum_{m=0}^{N-1} \left[\sum_{\ell=0}^m \binom{\ell+N-1}{\ell} \binom{N-1-\ell}{m-\ell} \right] (x-a)^m (b-x)^{2N-1-m}. \end{aligned}$$

By using the elementary identity $\sum_{\ell=0}^n \binom{\ell+p}{\ell} \binom{n+q-\ell}{n-\ell} = \binom{n+p+q+1}{n}$ which comes from the equality $(1+u)^{-p}(1+u)^{-q} = (1+u)^{-p-q}$ together with the expansion, e.g., for p , $(1+u)^{-p} = \sum_{\ell=0}^{\infty} (-1)^\ell \binom{p-1}{\ell} u^\ell$, we get that

$$\sum_{\ell=0}^m \binom{\ell+N-1}{\ell} \binom{N-1-\ell}{m-\ell} = \binom{2N-1}{m}.$$

As a byproduct,

$$H_0^-(x) = \frac{1}{(b-a)^{2N-1}} \sum_{m=0}^{N-1} \binom{2N-1}{m} (x-a)^m (b-x)^{2N-1-m}.$$

Similarly,

$$H_0^+(x) = \frac{1}{(b-a)^{2N-1}} \sum_{m=N}^{2N-1} \binom{2N-1}{m} (x-a)^m (b-x)^{2N-1-m}$$

and we immediately deduce that

$$\mathbb{P}\{\tau_{ab} < +\infty\} = H_0^-(x) + H_0^+(x) = \frac{1}{(b-a)^{2N-1}} \sum_{m=0}^{2N-1} \binom{2N-1}{m} (x-a)^m (b-x)^{2N-1-m} = 1.$$

□

Let us introduce the first down- and up-overshooting times of the single thresholds a and b for $(X_t)_{t \geq 0}$:

$$\tau_a^- = \inf\{t \geq 0 : X_t < a\}, \quad \tau_b^+ = \inf\{t \geq 0 : X_t > b\}.$$

The famous problem of the ruin of the gambler in the context of pseudo-Brownian motion consists in computing the pseudo-probability of overshooting one level (a or b) before the other one. For instance, we have that

$$\mathbb{P}_x\{\tau_a^- < \tau_b^+\} = \mathbb{P}_x\{X_{\tau_{ab}} \leq a\}.$$

Hence, in view of formula (5.1), we obtain the following result.

Corollary 3. *The “ruin” pseudo-probabilities related to pseudo-Brownian motion are given by*

$$\mathbb{P}_x\{\tau_a^- < \tau_b^+\} = H_0^-(x), \quad \mathbb{P}_x\{\tau_b^+ < \tau_a^-\} = H_0^+(x).$$

In the corollary below, we provide a way for computing the pseudo-moments of $X_{\tau_{ab}}$.

Corollary 4. *Let P be a polynomial and R the remainder of the Euclidean division of $P(x)$ by $(x-a)^N(x-b)^N$. We have that*

$$\mathbb{E}_x[P(X_{\tau_{ab}})] = R(x).$$

In particular, the pseudo-moments of $X_{\tau_{ab}}$ are given, for any $p \in \{0, 1, \dots, 2N-1\}$, by

$$\mathbb{E}_x[(X_{\tau_{ab}})^p] = x^p$$

and for any positive integer p , by setting $c_n = \sum_{k=0}^n \binom{N+k-1}{k} \binom{N+n-1-k}{n-k} a^k b^{n-k}$, by

$$\begin{aligned} \mathbb{E}_x[(X_{\tau_{ab}})^{2N+p}] &= x^{2N+p} - \left(\sum_{n=0}^p c_{p-n} x^n \right) (x-a)^N (x-b)^N \\ &= x^{2N+p} - \left[x^p + N(a+b) x^{p-1} \right. \\ &\quad \left. + \left(\frac{1}{2} N(N+1)(a^2 + b^2) + N^2 ab \right) x^{p-2} + \dots \right] (x-a)^N (x-b)^N. \end{aligned}$$

For instance,

$$\begin{aligned} \mathbb{E}_x[(X_{\tau_{ab}})^{2N}] &= x^{2N} - (x-a)^N (x-b)^N, \\ \mathbb{E}_x[(X_{\tau_{ab}})^{2N+1}] &= x^{2N+1} - [x + N(a+b)] (x-a)^N (x-b)^N, \\ \mathbb{E}_x[(X_{\tau_{ab}})^{2N+2}] &= x^{2N+2} - \left[x^2 + N(a+b)x + \left(\frac{1}{2} N(N+1)(a^2 + b^2) + N^2 ab \right) \right] (x-a)^N (x-b)^N. \end{aligned}$$

Proof. Let us introduce the quotient Q of the Euclidean division of $P(x)$ by $(x-a)^N(x-b)^N$: we have $P(x) = Q(x)(x-a)^N(x-b)^N + R(x)$. The polynomial R is of degree not greater than $(2N-1)$. Since a and b are roots of the polynomial $P(x) - R(x) = Q(x)(x-a)^N(x-b)^N$ with a

multiplicity not less than N , the successive derivatives of $P - R$ up to order $(N - 1)$ vanish at a and b . Therefore, by (5.2), we deduce that

$$\mathbb{E}_x \left[Q(X_{\tau_{ab}})(X_{\tau_{ab}} - a)^N (X_{\tau_{ab}} - b)^N \right] = 0$$

and then

$$\mathbb{E}_x[P(X_{\tau_{ab}})] = \mathbb{E}_x[R(X_{\tau_{ab}})].$$

Since the polynomial R is of degree not greater than $(2N - 1)$, we can write the decomposition

$$R(x) = \sum_{k=0}^{N-1} R^{(k)}(a) H_k^-(x) + \sum_{k=0}^{N-1} R^{(k)}(b) H_k^+(x).$$

Therefore, appealing to Remark 3, we obtain that

$$\mathbb{E}_x[P(X_{\tau_{ab}})] = \mathbb{E}_x[R(X_{\tau_{ab}})] = R(x) = P(x) - Q(x)(x - a)^N(x - b)^N.$$

Next, we compute the quotient Q when $P(x) = x^{2N+p}$:

$$\begin{aligned} \frac{x^{2N+p}}{(x-a)^N(x-b)^N} &= x^p \left(1 - \frac{a}{x}\right)^{-N} \left(1 - \frac{b}{x}\right)^{-N} \\ &= x^p \left(\sum_{k=0}^{\infty} \binom{N+k-1}{k} \frac{a^k}{x^k} \right) \left(\sum_{\ell=0}^{\infty} \binom{N+\ell-1}{\ell} \frac{b^\ell}{x^\ell} \right) \\ &= \sum_{k=0}^{\infty} c_n x^{p-n} = \sum_{k=0}^p c_{p-n} x^n + \sum_{n=1}^{\infty} \frac{c_{n+p}}{x^n} \end{aligned}$$

where

$$c_n = \sum_{\substack{k, \ell \geq 0 \\ k+\ell=n}} \binom{N+k-1}{k} \binom{N+\ell-1}{\ell} a^k b^\ell = \sum_{k=0}^n \binom{N+k-1}{k} \binom{N+n-1-k}{n-k} a^k b^{n-k}.$$

Then, the quotient of x^{2N+p} by $(x-a)^N(x-b)^N$ is equal to $\sum_{k=0}^n c_n x^{p-n}$. In particular,

$$c_0 = 1, \quad c_1 = N(a+b), \quad c_2 = \frac{1}{2} N(N+1)(a^2 + b^2) + N^2 ab.$$

□

Remark 5. By (5.2), we easily get that

$$\mathbb{E}_x \left[(X_{\tau_{ab}} - b)^p \mathbf{1}_{\{\tau_b^+ < \tau_a^-\}} \right] = \begin{cases} p! H_p^+(x) & \text{if } p \leq N, \\ 0 & \text{if } p \geq N+1. \end{cases}$$

This formula suggests the following interpretation of Hermite polynomials in terms of pseudo-Brownian motion: for $p \in \{0, \dots, N-1\}$,

$$H_p^+(x) = \frac{1}{p!} \mathbb{E}_x \left[(X_{\tau_{ab}} - b)^p \mathbf{1}_{\{\tau_b^+ < \tau_a^-\}} \right].$$

6 The case $N = 2$

For $N = 2$, pseudo-Brownian motion is the so-called biharmonic-pseudo-process. In this case, the settings write $\theta_1 = e^{i3\pi/4}$, $\theta_2 = e^{i5\pi/4} = \bar{\theta}_1$, $\theta_3 = e^{i7\pi/4}$, $\theta_4 = e^{i\pi/4} = \bar{\theta}_3$, and, by setting $\nu = \lambda/4$,

$$\Delta(\lambda) = \begin{vmatrix} e^{\theta_1 a} & e^{\theta_2 a} & e^{\theta_3 a} & e^{\theta_4 a} \\ \theta_1 e^{\theta_1 a} & \theta_2 e^{\theta_2 a} & \theta_3 e^{\theta_3 a} & \theta_4 e^{\theta_4 a} \\ e^{\theta_1 b} & e^{\theta_2 b} & e^{\theta_3 b} & e^{\theta_4 b} \\ \theta_1 e^{\theta_1 b} & \theta_2 e^{\theta_2 b} & \theta_3 e^{\theta_3 b} & \theta_4 e^{\theta_4 b} \end{vmatrix},$$

$$\Delta_0^-(\lambda; x) = \begin{vmatrix} e_\lambda^{\theta_1 x} & e_\lambda^{\theta_2 x} & e_\lambda^{\theta_3 x} & e_\lambda^{\theta_4 x} \\ \theta_1 e_\lambda^{\theta_1 a} & \theta_2 e_\lambda^{\theta_2 a} & \theta_3 e_\lambda^{\theta_3 a} & \theta_4 e_\lambda^{\theta_4 a} \\ e_\lambda^{\theta_1 b} & e_\lambda^{\theta_2 b} & e_\lambda^{\theta_3 b} & e_\lambda^{\theta_4 b} \\ \theta_1 e_\lambda^{\theta_1 b} & \theta_2 e_\lambda^{\theta_2 b} & \theta_3 e_\lambda^{\theta_3 b} & \theta_4 e_\lambda^{\theta_4 b} \end{vmatrix}, \quad \Delta_1^-(\lambda; x) = \begin{vmatrix} e_\lambda^{\theta_1 a} & e_\lambda^{\theta_2 a} & e_\lambda^{\theta_3 a} & e_\lambda^{\theta_4 a} \\ e_\lambda^{\theta_1 x} & e_\lambda^{\theta_2 x} & e_\lambda^{\theta_3 x} & e_\lambda^{\theta_4 x} \\ e_\lambda^{\theta_1 b} & e_\lambda^{\theta_2 b} & e_\lambda^{\theta_3 b} & e_\lambda^{\theta_4 b} \\ \theta_1 e_\lambda^{\theta_1 b} & \theta_2 e_\lambda^{\theta_2 b} & \theta_3 e_\lambda^{\theta_3 b} & \theta_4 e_\lambda^{\theta_4 b} \end{vmatrix},$$

$$\Delta_0^+(\lambda; x) = \begin{vmatrix} e_\lambda^{\theta_1 a} & e_\lambda^{\theta_2 a} & e_\lambda^{\theta_3 a} & e_\lambda^{\theta_4 a} \\ \theta_1 e_\lambda^{\theta_1 a} & \theta_2 e_\lambda^{\theta_2 a} & \theta_3 e_\lambda^{\theta_3 a} & \theta_4 e_\lambda^{\theta_4 a} \\ e_\lambda^{\theta_1 x} & e_\lambda^{\theta_2 x} & e_\lambda^{\theta_3 x} & e_\lambda^{\theta_4 x} \\ \theta_1 e_\lambda^{\theta_1 b} & \theta_2 e_\lambda^{\theta_2 b} & \theta_3 e_\lambda^{\theta_3 b} & \theta_4 e_\lambda^{\theta_4 b} \end{vmatrix}, \quad \Delta_1^+(\lambda; x) = \begin{vmatrix} e_\lambda^{\theta_1 a} & e_\lambda^{\theta_2 a} & e_\lambda^{\theta_3 a} & e_\lambda^{\theta_4 a} \\ \theta_1 e_\lambda^{\theta_1 a} & \theta_2 e_\lambda^{\theta_2 a} & \theta_3 e_\lambda^{\theta_3 a} & \theta_4 e_\lambda^{\theta_4 a} \\ e_\lambda^{\theta_1 b} & e_\lambda^{\theta_2 b} & e_\lambda^{\theta_3 b} & e_\lambda^{\theta_4 b} \\ e_\lambda^{\theta_1 x} & e_\lambda^{\theta_2 x} & e_\lambda^{\theta_3 x} & e_\lambda^{\theta_4 x} \end{vmatrix}.$$

Elementary computations yield that

$$\Delta(\lambda) = 4 [\cosh(2\sqrt[4]{v}(b-a)) + \cos(2\sqrt[4]{v}(b-a)) - 2].$$

Let us expand, e.g., $\Delta_0^-(\lambda; x)$ with respect to its first row:

$$\Delta_0^-(\lambda; x) = c_1 e_\lambda^{\theta_1 x} + c_2 e_\lambda^{\theta_2 x} + c_3 e_\lambda^{\theta_3 x} + c_4 e_\lambda^{\theta_4 x}$$

where c_1, c_2, c_3, c_4 are the cofactors of $\Delta_0^-(\lambda; x)$ related to the first row. Straightforward (but cumbersome) computations yield that $c_2 = \bar{c}_1$ and $c_4 = \bar{c}_3$ and

$$c_1 = (1-i) e^{\frac{4}{\sqrt[4]{v}}((2b-a)-ia)} + (1+i) e^{\frac{4}{\sqrt[4]{v}}(a-i(2b-a))} - 2 e^{(1-i)\frac{4}{\sqrt[4]{v}}a},$$

$$c_3 = (1-i) e^{-\frac{4}{\sqrt[4]{v}}((2b-a)-ia)} + (1+i) e^{-\frac{4}{\sqrt[4]{v}}(a-i(2b-a))} - 2 e^{-(1-i)\frac{4}{\sqrt[4]{v}}a}.$$

Therefore, we have that

$$\begin{aligned} \Delta_0^-(\lambda; x) &= 2 \Re(c_1 e_\lambda^{\theta_1 x} + c_3 e_\lambda^{\theta_3 x}) \\ &= 2 \left[e^{\frac{4}{\sqrt[4]{v}}(x-a)} \cos(\sqrt[4]{v}(x+a-2b)) + e^{-\frac{4}{\sqrt[4]{v}}(x-a)} \cos(\sqrt[4]{v}(x+a-2b)) \right. \\ &\quad + e^{\frac{4}{\sqrt[4]{v}}(x-a)} \sin(\sqrt[4]{v}(x+a-2b)) - e^{-\frac{4}{\sqrt[4]{v}}(x-a)} \sin(\sqrt[4]{v}(x+a-2b)) \\ &\quad - 2 e^{\frac{4}{\sqrt[4]{v}}(x-a)} \cos(\sqrt[4]{v}(x-a)) - 2 e^{-\frac{4}{\sqrt[4]{v}}(x-a)} \cos(\sqrt[4]{v}(x-a)) \\ &\quad + e^{\frac{4}{\sqrt[4]{v}}(x+a-2b)} \cos(\sqrt[4]{v}(x-a)) + e^{-\frac{4}{\sqrt[4]{v}}(x+a-2b)} \cos(\sqrt[4]{v}(x-a)) \\ &\quad \left. - e^{\frac{4}{\sqrt[4]{v}}(x+a-2b)} \sin(\sqrt[4]{v}(x-a)) + e^{-\frac{4}{\sqrt[4]{v}}(x+a-2b)} \sin(\sqrt[4]{v}(x-a)) \right] \end{aligned}$$

which simplifies by means of hyperbolic functions into

$$\begin{aligned} \Delta_0^-(\lambda; x) &= 4 [\cosh(\sqrt[4]{v}(x-a)) \cos(\sqrt[4]{v}(x+a-2b)) + \sinh(\sqrt[4]{v}(x-a)) \sin(\sqrt[4]{v}(x+a-2b)) \\ &\quad + \cosh(\sqrt[4]{v}(x+a-2b)) \cos(\sqrt[4]{v}(x-a)) - \sinh(\sqrt[4]{v}(x+a-2b)) \sin(\sqrt[4]{v}(x-a)) \\ &\quad - 2 \cosh(\sqrt[4]{v}(x-a)) \cos(\sqrt[4]{v}(x-a))]. \end{aligned}$$

Quite similar computations yield that

$$\begin{aligned} \Delta_1^-(\lambda; x) &= 4 [\cosh(\sqrt[4]{v}(x+a-2b)) \sin(\sqrt[4]{v}(x-a)) + \sinh(\sqrt[4]{v}(x-a)) \cos(\sqrt[4]{v}(x+a-2b)) \\ &\quad - \cosh(\sqrt[4]{v}(x-a)) \sin(\sqrt[4]{v}(x-a)) - \sinh(\sqrt[4]{v}(x-a)) \cos(\sqrt[4]{v}(x-a))]. \end{aligned}$$

The determinants Δ_0^+ and Δ_1^+ can be immediately deduced from Δ_0^- and Δ_1^- by interchanging the roles of a and b as it can be seen upon interchanging certain rows therein. We obtain that

$$\begin{aligned} \Delta_0^+(\lambda; x) &= 4 [\cosh(\sqrt[4]{v}(x-b)) \cos(\sqrt[4]{v}(x+b-2a)) + \sinh(\sqrt[4]{v}(x-b)) \sin(\sqrt[4]{v}(x+b-2a)) \\ &\quad + \cosh(\sqrt[4]{v}(x+b-2a)) \cos(\sqrt[4]{v}(x-b)) - \sinh(\sqrt[4]{v}(x+b-2a)) \sin(\sqrt[4]{v}(x-b)) \\ &\quad - 2 \cosh(\sqrt[4]{v}(x-b)) \cos(\sqrt[4]{v}(x-b))], \\ \Delta_1^+(\lambda; x) &= 4 [\cosh(\sqrt[4]{v}(x+b-2a)) \sin(\sqrt[4]{v}(x-b)) + \sinh(\sqrt[4]{v}(x-b)) \cos(\sqrt[4]{v}(x+b-2a)) \\ &\quad - \cosh(\sqrt[4]{v}(x-b)) \sin(\sqrt[4]{v}(x-b)) - \sinh(\sqrt[4]{v}(x-b)) \cos(\sqrt[4]{v}(x-b))]. \end{aligned}$$

Now, formula (3.5) reads

$$\mathbb{P}_x\{\tau_{ab} \in dt, X_{\tau_{ab}} \in dz\} / dt dz = I_0^-(t; x) \delta_a(z) + I_1^-(t; x) \delta'_a(z) + I_0^+(t; x) \delta_b(z) + I_1^+(t; x) \delta'_b(z)$$

where the functions I_0^\pm and I_1^\pm are characterized by

$$\int_0^\infty I_0^\pm(t; x) e^{-\lambda t} dt = \frac{\Delta_0^\pm(\lambda; x)}{\Delta(\lambda)}, \quad \int_0^\infty I_1^\pm(t; x) e^{-\lambda t} dt = \frac{1}{\sqrt[4]{\lambda}} \frac{\Delta_1^\pm(\lambda; x)}{\Delta(\lambda)}.$$

Concerning the pseudo-distribution of the exit location $X_{\tau_{ab}}$, it is given by

$$\mathbb{P}_x\{X_{\tau_{ab}} \in dz\} / dz = H_0^-(x) \delta_a(z) - H_1^-(x) \delta'_a(z) + H_0^+(x) \delta_b(z) - H_1^+(x) \delta'_b(z)$$

with

$$\begin{aligned} H_0^-(x) &= \frac{(x-b)^2(2x-3a+b)}{(b-a)^3}, & H_1^-(x) &= \frac{(x-a)(x-b)^2}{(b-a)^2}, \\ H_0^+(x) &= -\frac{(x-a)^2(2x+a-3b)}{(b-a)^3}, & H_1^+(x) &= \frac{(x-a)^2(x-b)}{(b-a)^2}. \end{aligned}$$

When the pseudo-process starts at the middle of the interval $[a, b]$, we obtain the following expressions for the determinants of interest: by setting $L = (b-a)/2$,

$$\Delta(\lambda) = 32 \left[\cosh^2(\sqrt[4]{\nu} L) \sinh^2(\sqrt[4]{\nu} L) - \cos^2(\sqrt[4]{\nu} L) \sin^2(\sqrt[4]{\nu} L) \right],$$

$$\begin{aligned} \Delta_0^-\left(\lambda; \frac{a+b}{2}\right) &= \Delta_0^+\left(\lambda; \frac{a+b}{2}\right) = 4 \left[\cosh(\sqrt[4]{\nu} L) \cos(\sqrt[4]{\nu} L) \left(\cosh^2(\sqrt[4]{\nu} L) + \cos^2(\sqrt[4]{\nu} L) - 2 \right) \right. \\ &\quad \left. + \sinh(\sqrt[4]{\nu} L) \sin(\sqrt[4]{\nu} L) \left(\cosh^2(\sqrt[4]{\nu} L) - \cos^2(\sqrt[4]{\nu} L) \right) \right], \\ \Delta_1^-\left(\lambda; \frac{a+b}{2}\right) &= -\Delta_1^+\left(\lambda; \frac{a+b}{2}\right) = 4 \left[\cosh(\sqrt[4]{\nu} L) \sin(\sqrt[4]{\nu} L) \sinh^2(\sqrt[4]{\nu} L) \right. \\ &\quad \left. - \sinh(\sqrt[4]{\nu} L) \cos(\sqrt[4]{\nu} L) \sin^2(\sqrt[4]{\nu} L) \right]. \end{aligned}$$

Hence, in this case, we have the following symmetric expression:

$$\mathbb{P}_{\frac{a+b}{2}}\{\tau_{ab} \in dt, X_{\tau_{ab}} \in dz\} / dt dz = I_0^+\left(t; \frac{a+b}{2}\right) (\delta_a(z) + \delta_b(z)) + I_1^+\left(t; \frac{a+b}{2}\right) (\delta'_b(z) - \delta'_a(z)).$$

Moreover,

$$H_0^-\left(\lambda; \frac{a+b}{2}\right) = H_0^+\left(\lambda; \frac{a+b}{2}\right) = \frac{1}{2}, \quad H_1^-\left(\lambda; \frac{a+b}{2}\right) = -H_1^+\left(\lambda; \frac{a+b}{2}\right) = \frac{L}{4}.$$

Then,

$$\mathbb{P}_{\frac{a+b}{2}}\{X_{\tau_{ab}} \in dz\} / dz = \frac{1}{2} (\delta_a(z) + \delta_b(z)) + \frac{b-a}{8} (\delta'_b(z) - \delta'_a(z)).$$

A Appendix

A.1 Asymptotics of $\Delta(\lambda)$ and $\Delta_k^\pm(\lambda; x)$ as b tends to $+\infty$

In this appendix, we check limits (3.6). By factorizing the ℓ th column of the determinant $\Delta(\lambda)$ by $e_\lambda^{\theta_\ell a}$ for each $\ell \in \{1, \dots, 2N\}$ and observing that $\sum_{\ell=1}^{2N} \theta_\ell = 0$, we find that

$$\Delta(\lambda) = \begin{vmatrix} 1 & \cdots & 1 \\ \theta_1 & \cdots & \theta_{2N} \\ \vdots & & \vdots \\ \theta_1^{N-1} & \cdots & \theta_{2N}^{N-1} \\ \cdots & \cdots & \cdots \\ e_\lambda^{\theta_1(b-a)} & \cdots & e_\lambda^{\theta_{2N}(b-a)} \\ \theta_1 e_\lambda^{\theta_1(b-a)} & \cdots & \theta_{2N} e_\lambda^{\theta_{2N}(b-a)} \\ \vdots & & \vdots \\ \theta_1^{N-1} e_\lambda^{\theta_1(b-a)} & \cdots & \theta_{2N}^{N-1} e_\lambda^{\theta_{2N}(b-a)} \end{vmatrix}.$$

We separate $\Delta(\lambda)$ into four squared blocks as follows:

$$\Delta(\lambda) = \begin{vmatrix} V & \vdots & \tilde{V} \\ \cdots & \cdots & \cdots \\ W(\lambda) & \vdots & \tilde{W}(\lambda) \end{vmatrix}$$

with

$$V = \begin{pmatrix} 1 & \cdots & 1 \\ \theta_1 & \cdots & \theta_N \\ \vdots & & \vdots \\ \theta_1^{N-1} & \cdots & \theta_N^{N-1} \end{pmatrix}, \quad \tilde{V} = \begin{pmatrix} 1 & \cdots & 1 \\ \theta_{N+1} & \cdots & \theta_{2N} \\ \vdots & & \vdots \\ \theta_{N+1}^{N-1} & \cdots & \theta_{2N}^{N-1} \end{pmatrix},$$

$$W(\lambda) = \begin{pmatrix} e_\lambda^{\theta_1(b-a)} & \cdots & e_\lambda^{\theta_N(b-a)} \\ \theta_1 e_\lambda^{\theta_1(b-a)} & \cdots & \theta_N e_\lambda^{\theta_N(b-a)} \\ \vdots & & \vdots \\ \theta_1^{N-1} e_\lambda^{\theta_1(b-a)} & \cdots & \theta_N^{N-1} e_\lambda^{\theta_N(b-a)} \end{pmatrix},$$

$$\tilde{W}(\lambda) = \begin{pmatrix} e_\lambda^{\theta_{N+1}(b-a)} & \cdots & e_\lambda^{\theta_{2N}(b-a)} \\ \theta_{N+1} e_\lambda^{\theta_{N+1}(b-a)} & \cdots & \theta_{2N} e_\lambda^{\theta_{2N}(b-a)} \\ \vdots & & \vdots \\ \theta_{N+1}^{N-1} e_\lambda^{\theta_{N+1}(b-a)} & \cdots & \theta_{2N}^{N-1} e_\lambda^{\theta_{2N}(b-a)} \end{pmatrix}.$$

Due to the fact that $\Re(\theta_\ell) < 0$ for $\ell \in \{1, \dots, N\}$ and $\Re(\theta_\ell) > 0$ for $\ell \in \{N+1, \dots, 2N\}$, it may be easily seen by using an expansion by blocks of type $N \times N$ that the leading terms of $\Delta(\lambda)$ are obtained by performing the product of the determinants of both diagonal blocks V and $\tilde{W}(\lambda)$, namely:

$$\Delta(\lambda) \underset{b \rightarrow +\infty}{\sim} \det(V) \times \det(\tilde{W}(\lambda)).$$

Similarly, we decompose $\Delta_k^-(\lambda; x)$ into

$$\Delta_k^-(\lambda; x) = \begin{vmatrix} V_k(\lambda; x) & \vdots & \tilde{V}_k(\lambda; x) \\ \cdots & \cdots & \cdots \\ W(\lambda) & \vdots & \tilde{W}(\lambda) \end{vmatrix}$$

with

$$V_k(\lambda; x) = \begin{pmatrix} 1 & \cdots & 1 \\ \theta_1 & \cdots & \theta_N \\ \vdots & & \vdots \\ \theta_1^{k-1} & \cdots & \theta_N^{k-1} \\ e_\lambda^{\theta_1(x-a)} & \cdots & e_\lambda^{\theta_N(x-a)} \\ \theta_1^{k+1} & \cdots & \theta_N^{k+1} \\ \vdots & & \vdots \\ \theta_1^{N-1} & \cdots & \theta_N^{N-1} \end{pmatrix}, \quad \tilde{V}_k(\lambda; x) = \begin{pmatrix} 1 & \cdots & 1 \\ \theta_{N+1} & \cdots & \theta_{2N} \\ \vdots & & \vdots \\ \theta_{N+1}^{k-1} & \cdots & \theta_{2N}^{k-1} \\ e_\lambda^{\theta_{N+1}(x-a)} & \cdots & e_\lambda^{\theta_{2N}(x-a)} \\ \theta_{N+1}^{k+1} & \cdots & \theta_{2N}^{k+1} \\ \vdots & & \vdots \\ \theta_{N+1}^{N-1} & \cdots & \theta_{2N}^{N-1} \end{pmatrix}.$$

We can easily see that, for $x \in (a, b)$,

$$\Delta_k^-(\lambda; x) \underset{b \rightarrow +\infty}{\sim} \det(V_k(\lambda; x)) \times \det(\tilde{W}(\lambda)).$$

As a byproduct, we get the first limit of (3.6):

$$\frac{\Delta_k^-(\lambda; x)}{\Delta(\lambda)} \xrightarrow{b \rightarrow +\infty} \frac{\det(V_k(\lambda; x))}{\det(V)}.$$

By expanding the determinant of $V_k(\lambda; x)$ with respect to its k th row, we obtain that

$$\frac{\det(V_k(\lambda; x))}{\det(V)} = \sum_{\ell=1}^N \alpha_{k\ell} e_\lambda^{\theta_\ell(x-a)}$$

where the coefficients $\alpha_{k\ell}$ are explicitly written in Remark 2.

Next, concerning the determinant $\Delta_k^+(\lambda; x)$, by factorizing the ℓ th column by $e_\lambda^{\theta_\ell b}$ for each $\ell \in \{1, \dots, 2N\}$, using the identity $\sum_{\ell=1}^{2N} \theta_\ell = 0$ and permuting the k th and $(N+k)$ th rows for each $k \in \{1, \dots, N\}$, we get that

$$\Delta_k^+(\lambda; x) = (-1)^N \begin{vmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ \theta_1^{k-1} & \cdots & \theta_{2N}^{k-1} \\ e_\lambda^{\theta_1(x-b)} & \cdots & e_\lambda^{\theta_{2N}(x-b)} \\ \theta_1^{k+1} & \cdots & \theta_{2N}^{k+1} \\ \vdots & & \vdots \\ \theta_1^{N-1} & \cdots & \theta_{2N}^{N-1} \\ \cdots & & \cdots \\ e_\lambda^{\theta_1(a-b)} & \cdots & e_\lambda^{\theta_{2N}(a-b)} \\ \vdots & & \vdots \\ \theta_1^{N-1} e_\lambda^{\theta_1(a-b)} & \cdots & \theta_{2N}^{N-1} e_\lambda^{\theta_{2N}(a-b)} \end{vmatrix}.$$

As previously, we decompose $\Delta_k^+(\lambda; x)$ into

$$\Delta_k^+(\lambda; x) = (-1)^N \begin{vmatrix} Y_k(\lambda; x) & \vdots & \tilde{Y}_k(\lambda; x) \\ \cdots & & \cdots \\ Z(\lambda) & \vdots & \tilde{Z}(\lambda) \end{vmatrix} = \begin{vmatrix} \tilde{Y}_k(\lambda; x) & \vdots & Y_k(\lambda; x) \\ \cdots & & \cdots \\ \tilde{Z}(\lambda) & \vdots & Z(\lambda) \end{vmatrix}$$

with

$$Y_k(\lambda; x) = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ \theta_1^{k-1} & \cdots & \theta_N^{k-1} \\ e_\lambda^{\theta_1(x-b)} & \cdots & e_\lambda^{\theta_N(x-b)} \\ \theta_1^{k+1} & \cdots & \theta_N^{k+1} \\ \vdots & & \vdots \\ \theta_1^{N-1} & \cdots & \theta_N^{N-1} \end{pmatrix}, \quad \tilde{Y}_k(\lambda; x) = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & & \vdots \\ \theta_{N+1}^{k-1} & \cdots & \theta_{2N}^{k-1} \\ e_\lambda^{\theta_{N+1}(x-b)} & \cdots & e_\lambda^{\theta_{2N}(x-b)} \\ \theta_{N+1}^{k+1} & \cdots & \theta_{2N}^{k+1} \\ \vdots & & \vdots \\ \theta_{N+1}^{N-1} & \cdots & \theta_{2N}^{N-1} \end{pmatrix},$$

$$Z(\lambda) = \begin{pmatrix} e_\lambda^{\theta_1(a-b)} & \cdots & e_\lambda^{\theta_N(a-b)} \\ \theta_1 e_\lambda^{\theta_1(a-b)} & \cdots & \theta_N e_\lambda^{\theta_N(a-b)} \\ \vdots & & \vdots \\ \theta_1^{N-1} e_\lambda^{\theta_1(a-b)} & \cdots & \theta_N^{N-1} e_\lambda^{\theta_N(a-b)} \end{pmatrix},$$

$$\tilde{Z}(\lambda) = \begin{pmatrix} e_\lambda^{\theta_{N+1}(a-b)} & \cdots & e_\lambda^{\theta_{2N}(a-b)} \\ \theta_{N+1} e_\lambda^{\theta_{N+1}(a-b)} & \cdots & \theta_{2N} e_\lambda^{\theta_{2N}(a-b)} \\ \vdots & & \vdots \\ \theta_{N+1}^{N-1} e_\lambda^{\theta_{N+1}(a-b)} & \cdots & \theta_{2N}^{N-1} e_\lambda^{\theta_{2N}(a-b)} \end{pmatrix}.$$

Finally, by remarking that $\theta_{N+\ell} = -\theta_\ell$ for any $\ell \in \{1, \dots, N\}$, we derive that

$$\Delta_k^+(\lambda; x) = (-1)^k \begin{vmatrix} U_k(\lambda; x) & \tilde{U}_k(\lambda; x) \\ \vdots & \vdots \\ W(\lambda) & \tilde{W}(\lambda) \end{vmatrix}$$

where the matrices $U_k(\lambda; x)$ and $\tilde{U}_k(\lambda; x)$ are deduced from $V_k(\lambda; x)$ and $\tilde{V}_k(\lambda; x)$ by changing $(x - a)$ into $(b - x)$, that is, $U_k(\lambda; x) = V_k(\lambda; a + b - x)$ and $\tilde{U}_k(\lambda; x) = \tilde{V}_k(\lambda; a + b - x)$. As a byproduct, we derive the identity

$$\Delta_k^+(\lambda; x) = (-1)^k \Delta_k^-(\lambda; a + b - x) \quad (\text{A.1})$$

which is evoked in Remark 1. Thanks to an expansion by blocks, we can see that, for $x \in (a, b)$,

$$\Delta_k^+(\lambda; x) \underset{b \rightarrow +\infty}{\sim} (-1)^k \det(V_k(\lambda; a + b - x)) \times \det(\tilde{W}(\lambda)) = o[\det(\tilde{W}(\lambda))].$$

From this, we deduce the second limit of (3.6):

$$\frac{\Delta_k^+(\lambda; x)}{\Delta(\lambda)} \underset{b \rightarrow +\infty}{\longrightarrow} 0.$$

A.2 Asymptotics of $\Delta(\lambda)$ and $\Delta_k^\pm(\lambda; x)$ as λ tends to 0^+ or $+\infty$

The procedure depicted in the previous subparagraph can be carried out *mutatis mutandis* in the case where λ tends to $+\infty$. This yields the following limiting result:

$$\frac{\Delta_k^\pm(\lambda; x)}{\Delta(\lambda)} \underset{\lambda \rightarrow +\infty}{\longrightarrow} 0, \quad (\text{A.2})$$

the rate of convergence being exponential. Then $I_k^\pm(t; x) \underset{t \rightarrow 0^+}{\longrightarrow} 0$. Below, we examine the case where λ tends to 0^+ .

A.2.1 Asymptotics of $\Delta(\lambda)$ as λ tends to 0^+

Set $c = \lambda^{1/(2N)}(b-a)$. The number c tends to 0. We expand the exponentials lying in $\Delta(\lambda)$ into power series: for $k \in \{0, \dots, N-1\}$,

$$\theta_\ell^k e^{\theta_\ell(b-a)} = \sum_{i=0}^{\infty} \theta_\ell^{i+k} \frac{c^i}{i!} = \sum_{i=k}^{\infty} \theta_\ell^i \frac{c^{i-k}}{(i-k)!}.$$

Then,

$$\Delta(\lambda) = \begin{vmatrix} 1 & \dots & 1 \\ \theta_1 & \dots & \theta_{2N} \\ \vdots & & \vdots \\ \theta_1^{N-1} & \dots & \theta_{2N}^{N-1} \\ \hline \sum_{i=0}^{\infty} \theta_1^i \frac{c^i}{i!} & \dots & \sum_{i=0}^{\infty} \theta_{2N}^i \frac{c^i}{i!} \\ \sum_{i=1}^{\infty} \theta_1^i \frac{c^{i-1}}{(i-1)!} & \dots & \sum_{i=1}^{\infty} \theta_{2N}^i \frac{c^{i-1}}{(i-1)!} \\ \vdots & & \vdots \\ \sum_{i=N-1}^{\infty} \theta_1^i \frac{c^{i-N+1}}{(i-N+1)!} & \dots & \sum_{i=N-1}^{\infty} \theta_{2N}^i \frac{c^{i-N+1}}{(i-N+1)!} \end{vmatrix}.$$

By multilinearity, we see that the terms including a power of θ_ℓ less than N can be discarded (for these terms, the corresponding determinant has two or more identical rows, thus it vanishes). Hence, the determinant $\Delta(\lambda)$ does not change if we only keep the sums $\sum_{i=N}^{\infty} \theta_\ell^i c^{i-k} / (i-k)!$:

$$\Delta(\lambda) = \begin{vmatrix} 1 & \dots & 1 \\ \theta_1 & \dots & \theta_{2N} \\ \vdots & & \vdots \\ \theta_1^{N-1} & \dots & \theta_{2N}^{N-1} \\ \hline \sum_{i=N}^{\infty} \theta_1^i \frac{c^i}{i!} & \dots & \sum_{i=N}^{\infty} \theta_{2N}^i \frac{c^i}{i!} \\ \sum_{i=N}^{\infty} \theta_1^i \frac{c^{i-1}}{(i-1)!} & \dots & \sum_{i=N}^{\infty} \theta_{2N}^i \frac{c^{i-1}}{(i-1)!} \\ \vdots & & \vdots \\ \sum_{i=N}^{\infty} \theta_1^i \frac{c^{i-N+1}}{(i-N+1)!} & \dots & \sum_{i=N}^{\infty} \theta_{2N}^i \frac{c^{i-N+1}}{(i-N+1)!} \end{vmatrix}.$$

By multilinearity, we can rewrite $\Delta(\lambda)$ as

$$\Delta(\lambda) = \sum_{\substack{i_1, \dots, i_N \geq N \\ i_1, \dots, i_N \text{ all distinct}}} \frac{c^{i_1 + (i_2-1) + \dots + (i_N - N + 1)}}{i_1! (i_2-1)! \dots (i_N - N + 1)!} \begin{vmatrix} 1 & \dots & 1 \\ \theta_1 & \dots & \theta_{2N} \\ \vdots & & \vdots \\ \theta_1^{N-1} & \dots & \theta_{2N}^{N-1} \\ \hline \theta_1^{i_1} & \dots & \theta_{2N}^{i_1} \\ \theta_1^{i_2} & \dots & \theta_{2N}^{i_2} \\ \vdots & & \vdots \\ \theta_1^{i_N} & \dots & \theta_{2N}^{i_N} \end{vmatrix}.$$

Because of the conditions on the indices i_1, \dots, i_N , the least power of c is not less than N^2 : indeed, the indices being distinct and not less than N , we have $i_1 + i_2 + \dots + i_N \geq N + (N+1) + \dots + (2N-1)$ or, equivalently, $i_1 + (i_2-1) + \dots + (i_N - N + 1) \geq N^2$. Moreover, if an index is greater than $(2N-1)$, say $i_N \geq 2N$, then $i_1 + i_2 + \dots + i_{N-1} + i_N \geq N + (N+1) + \dots + (2N-2) + (2N)$, that is, $i_1 + (i_2-1) + \dots + (i_N - N + 1) \geq N^2 + 1$. In words, the term c^{N^2} is obtained at

most for the indices not greater than $(2N - 1)$. Consequently, we see that the terms of the sums corresponding to i greater than $(2N - 1)$ can be neglected when c tends to 0, namely:

$$\Delta(\lambda) \underset{c \rightarrow 0^+}{=} \begin{vmatrix} 1 & \cdots & 1 \\ \theta_1 & \cdots & \theta_{2N} \\ \vdots & & \vdots \\ \theta_1^{N-1} & \cdots & \theta_{2N}^{N-1} \\ \cdots & & \cdots \\ \sum_{i=N}^{2N-1} \theta_1^i \frac{c^i}{i!} & \cdots & \sum_{i=N}^{2N-1} \theta_{2N}^i \frac{c^i}{i!} \\ \sum_{i=N}^{2N-1} \theta_1^i \frac{c^{i-1}}{(i-1)!} & \cdots & \sum_{i=N}^{2N-1} \theta_{2N}^i \frac{c^{i-1}}{(i-1)!} \\ \vdots & & \vdots \\ \sum_{i=N}^{2N-1} \theta_1^i \frac{c^{i-N+1}}{(i-N+1)!} & \cdots & \sum_{i=N}^{2N-1} \theta_{2N}^i \frac{c^{i-N+1}}{(i-N+1)!} \end{vmatrix} + o(c^{N^2})$$

We observe that the matrix lying in the foregoing determinant can be factorized into the product of the two following matrices:

$$A_1 = \begin{pmatrix} I & \vdots & O \\ \cdots & \cdots & \cdots \\ O & \vdots & B \end{pmatrix}, \quad A_2 = \begin{pmatrix} 1 & \cdots & 1 \\ \theta_1 & \cdots & \theta_{2N} \\ \vdots & & \vdots \\ \theta_1^{2N-1} & \cdots & \theta_{2N}^{2N-1} \end{pmatrix}$$

where I and O are respectively the unit and zero matrices of type $N \times N$, and

$$B = \begin{pmatrix} \frac{c^N}{N!} & \frac{c^{N+1}}{(N+1)!} & \cdots & \frac{c^{2N-1}}{(2N-1)!} \\ \frac{c^{N-1}}{(N-1)!} & \frac{c^N}{N!} & \cdots & \frac{c^{2N-2}}{(2N-2)!} \\ \vdots & \vdots & & \vdots \\ c & \frac{c^2}{2!} & \cdots & \frac{c^N}{N!} \end{pmatrix}.$$

We can decompose B into $C_1 \tilde{B} C_2$ where C_1 and C_2 are the diagonal matrices with c^N, c^{N-1}, \dots, c and $1, c, \dots, c^{N-1}$ as diagonal terms respectively, and

$$\tilde{B} = \begin{pmatrix} \frac{1}{N!} & \frac{1}{(N+1)!} & \cdots & \frac{1}{(2N-1)!} \\ \frac{1}{(N-1)!} & \frac{1}{N!} & \cdots & \frac{1}{(2N-2)!} \\ \vdots & \vdots & & \vdots \\ 1 & \frac{1}{2!} & \cdots & \frac{1}{N!} \end{pmatrix}.$$

Hence, all this discussion plainly entails that

$$\Delta(\lambda) \underset{c \rightarrow 0^+}{\sim} \det(A_1) \times \det(A_2) = \det(A_2) \times \det(\tilde{B}) \times \det(C_1) \times \det(C_2) = \text{constant} \times c^{N^2}$$

where the constant does not vanish, or, by means of the variable λ ,

$$\Delta(\lambda) \underset{\lambda \rightarrow 0^+}{\sim} \text{constant} \times \lambda^{N/2}. \quad (\text{A.3})$$

A.2.2 Asymptotics of $\Delta_k^\pm(\lambda; x)$ as λ tends to 0^+

A similar analysis can be carried out in the case of the determinant $\Delta_k^\pm(\lambda; x)$. Recall that $c = \lambda^{1/(2N)}(b - a)$ and set $\gamma = \lambda^{1/(2N)}(x - a)$. The numbers c and γ tend to 0 as λ tends to 0^+ . E.g.,

for $\Delta_k^-(\lambda; x)$, we have that

$$\Delta_k^\pm(\lambda; x) = \begin{vmatrix} 1 & \cdots & 1 \\ \theta_1 & \cdots & \theta_{2N} \\ \vdots & & \vdots \\ \theta_1^{k-1} & \cdots & \theta_{2N}^{k-1} \\ \sum_{i=0}^{\infty} \theta_1^i \frac{\gamma^i}{i!} & \cdots & \sum_{i=0}^{\infty} \theta_{2N}^i \frac{\gamma^i}{i!} \\ \theta_1^{k+1} & \cdots & \theta_{2N}^{k+1} \\ \vdots & & \vdots \\ \theta_1^{N-1} & \cdots & \theta_{2N}^{N-1} \\ \hline \sum_{i=0}^{\infty} \theta_1^i \frac{c^i}{i!} & \cdots & \sum_{i=0}^{\infty} \theta_{2N}^i \frac{c^i}{i!} \\ \sum_{i=1}^{\infty} \theta_1^i \frac{c^{i-1}}{(i-1)!} & \cdots & \sum_{i=1}^{\infty} \theta_{2N}^i \frac{c^{i-1}}{(i-1)!} \\ \vdots & & \vdots \\ \sum_{i=N-1}^{\infty} \theta_1^i \frac{c^{i-N+1}}{(i-N+1)!} & \cdots & \sum_{i=N-1}^{\infty} \theta_{2N}^i \frac{c^{i-N+1}}{(i-N+1)!} \end{vmatrix}.$$

As previously, this determinant remains unchanged by removing the terms related to the indices $0, 1, \dots, k-1, k+1, \dots, N-1$ in each sum. Moreover, for obtaining an asymptotics when c, γ tend to 0 (actually c and γ have the same order of growth when λ tends to 0), it is enough to keep the terms related to the indices not greater than $(2N-1)$. Then, by setting $I_k = \{k\} \cup \{N, N+1, \dots, 2N-1\}$,

$$\Delta_k^-(\lambda; x)_{c, \gamma \rightarrow 0^+} = \begin{vmatrix} 1 & \cdots & 1 \\ \theta_1 & \cdots & \theta_{2N} \\ \vdots & & \vdots \\ \theta_1^{k-1} & \cdots & \theta_{2N}^{k-1} \\ \sum_{i \in I_k} \theta_1^i \frac{\gamma^i}{i!} & \cdots & \sum_{i \in I_k} \theta_{2N}^i \frac{\gamma^i}{i!} \\ \theta_1^{k+1} & \cdots & \theta_{2N}^{k+1} \\ \vdots & & \vdots \\ \theta_1^{N-1} & \cdots & \theta_{2N}^{N-1} \\ \hline \sum_{i \in I_k} \theta_1^i \frac{c^i}{i!} & \cdots & \sum_{i \in I_k} \theta_{2N}^i \frac{c^i}{i!} \\ \sum_{i \in I_k} \theta_1^i \frac{c^{i-1}}{(i-1)!} & \cdots & \sum_{i \in I_k} \theta_{2N}^i \frac{c^{i-1}}{(i-1)!} \\ \vdots & & \vdots \\ \sum_{i \in I_k} \theta_1^i \frac{c^{i-N+1}}{(i-N+1)!} & \cdots & \sum_{i \in I_k} \theta_{2N}^i \frac{c^{i-N+1}}{(i-N+1)!} \end{vmatrix} + o(\gamma^k c^{N^2}).$$

We observe that the matrix lying in the above determinant is the product of \tilde{A}_1 by A_2 where

- $\tilde{A}_1 = \begin{pmatrix} I_k & \vdots & O_{2,k} \\ \cdots & \cdots & \cdots \\ O_{1,k} & \vdots & B \end{pmatrix};$
- I_k is the diagonal matrix of type $N \times N$ with diagonal terms equal to 1 except for the $(k+1)$ th which is $\gamma^k/k!$;
- $O_{1,k}$ is the matrix of type $N \times N$ with all terms equal to 0 except for the $(k+1)$ th column which is made of $c^k/k!, c^{k-1}/(k-1)!, \dots, c, 1, 0, \dots, 0$;
- $O_{2,k}$ is the matrix of type $N \times N$ with all terms equal to 0 except for the $(k+1)$ th row which is made of $\gamma^N/N!, \gamma^{N+1}/(N+1)!, \dots, \gamma^{2N-1}/(2N-1)!$.

The determinant of \tilde{A}_1 remains unchanged by interchanging its $(k+1)$ th and N th columns and its $(k+1)$ th and $(k+1)$ th rows. This yields that

$$\det(\tilde{A}_1) = \begin{vmatrix} 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & \frac{\gamma^k}{k!} & \frac{\gamma^N}{N!} & \cdots & \frac{\gamma^{2N-1}}{(2N-1)!} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & \frac{c^k}{k!} & \frac{c^N}{N!} & \cdots & \frac{c^{2N-1}}{(2N-1)!} \\ 0 & \cdots & 0 & \frac{c^{k-1}}{(k-1)!} & \frac{c^{N-1}}{(N-1)!} & \cdots & \frac{c^{2N-2}}{(2N-2)!} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 1 & \frac{c^{N-k}}{(N-k)!} & \cdots & \frac{c^{2N-k-1}}{(2N-k-1)!} \\ 0 & \cdots & 0 & 0 & \frac{c^{N-k-1}}{(N-k-1)!} & \cdots & \frac{c^{2N-k-2}}{(2N-k-2)!} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & c & \cdots & \frac{c^N}{N!} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\gamma^k}{k!} & \frac{\gamma^N}{N!} & \cdots & \frac{\gamma^{2N-1}}{(2N-1)!} \\ \frac{c^k}{k!} & \frac{c^N}{N!} & \cdots & \frac{c^{2N-1}}{(2N-1)!} \\ \frac{c^{k-1}}{(k-1)!} & \frac{c^{N-1}}{(N-1)!} & \cdots & \frac{c^{2N-2}}{(2N-2)!} \\ \vdots & \vdots & & \vdots \\ 1 & \frac{c^{N-k}}{(N-k)!} & \cdots & \frac{c^{2N-k-1}}{(2N-k-1)!} \\ 0 & \frac{c^{N-k-1}}{(N-k-1)!} & \cdots & \frac{c^{2N-k-2}}{(2N-k-2)!} \\ \vdots & \vdots & & \vdots \\ 0 & c & \cdots & \frac{c^N}{N!} \end{vmatrix}.$$

By expanding this last determinant with respect to its first row, it is not difficult to see that $\det(\tilde{A}_1) = O(\gamma^k c^{N^2})$ (recall that c and γ have the same order of growth when λ tends to 0). Therefore, in terms of the variable λ ,

$$\Delta_k^-(\lambda; x) \underset{\lambda \rightarrow 0^+}{=} O(\lambda^{k/(2N)+N/2}) \quad (\text{A.4})$$

and the same holds for $\Delta_k^+(\lambda; x)$. Finally, by (A.3) and (A.4), we derive that

$$\frac{\Delta_k^-(\lambda; x)}{\Delta(\lambda)} \underset{\lambda \rightarrow 0^+}{=} O(\lambda^{k/(2N)}). \quad (\text{A.5})$$

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